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SOME THEORETICAL ASPECTS OF THE APPLICATION OF MATHEMATICAL MODELS IN ECONOMIC GEOGRAPHY

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It is the purpose of this paper to discuss some theoretical aspects of the application of mathematical models in economic geography. I do not intend to describe the possible range of mathematical models that can be used in the processes of scientific discovery in economic geography, but rather to consider some presuppositions and implications in the use of mathematical models. I think that a full understanding of the theoretical basis of mathematical models helps us to avoid some misunderstandings concerning their use.

As examples of mathematical models I shall use gravity and potential models. Whilst these models are in no way particularly sophisticated, they certainly constitute the simplest and at the same time the most widely used interaction models in economic geography. I hope that they will illustrate some general theoretical aspects of the use of mathematical models. There are many examples in the geographical literature of the use of mathematical models. The best analytical review is by P. Haggett (1965).

At this point it is necessary to consider the general meaning of a model, although this is not an easy task. The results of studies presented by methodologists show us that we cannot hope to give one unique structural definition for the concept of a model in different disciplines. Given that the model concept is variously interpreted and performs a variety of functions, the only way of reaching a definition of it is by way of formal pragmatics: system A is a model of system B if the study of A is useful for understanding B and if there is no direct or indirect interaction between A and B (L. Apostel, 1960, p. 180). The systems must therefore resemble one another and the resemblance is in terms of the pattern or order exhibited in each system. More specifically, the models have an isomorphic relationship, and the term analogue can be used as a generic term for both conceptual and physical isomorphs (A. Kaplan, 1964, p. 263).

R. L. Ackoff et al. (1962, p. 108) have suggested a simple three stage classification of models into iconic, analogue and symbolic models, in which each stage represents a higher degree of abstraction than the last. Iconic models represent properties at a different scale; analogue models represent one property by another; symbolic models represent properties by symbols. R. Chorley (1964) carried this classification process further and created a "model of models", illustrating it with examples from both physical and human geography.

Models can also be classified on the basis of their language. Physically a model is a non-linguistic system analogous to some other system being studied. Conceptually, if the model and its prototype are both general systems or structures one has the most abstract case of an algebraic model. If one of the terms of the relationship is represented by a language one has a semantic concept of a model (that is, relating a language to an arbitrary domain). If both terms are languages one has a syntactic concept (that is, relating two languages to each other). Very broadly speaking a mathematical model falls into the category of a semantic model.

The original field (e. g. the geographic reality we are concerned with) is thought of as being "projected" upon an abstract mathematical domain (made up of sets and functions defined on sets), such that the causal or structural relationships which hold in the original field can be said to be "modelled" by the relationships that hold in the mathematical domain. It is sometimes held that the mathematical model functions as a kind of ethereal analogue model, in which the mathematical equations are regarded as if they express some inherent property — an invisible mechanism — of the original system under investigation. This last suggestion has been rejected as an illusion by M. Black (1962, p. 224). A mathematical model has only one thing in common with analogues — the property of "as-if-ness". A mathematical model also rests on appeals to look at events "as if" the underlying causes or interactions had a certain structural analogy to the model proposed.

K. J. Arrow (1951) defines a mathematical model as a set of quantitative relationships expressed in the language of mathematics and describing the interaction of phenomena. Starting with the proposition that such a model consists of a system of equations, each of which in meant to measure in quantitative terms some distinct relationship, I must emphasize that this concept departs in many ways from the classical concept of the model as an isomorph. However, the two concepts have some similarity.

A mathematical model is really a simplification in mathematical language of a certain cognitive problem, so that this problem may be more readily understood and a solution attempted. Any mathematical model is an approximation of a given real-world situation. It is generally simpler than the situation it represents, for these situations are usually so complex that an exact representation would lead to futile mathematical complexity. Moreover, the simplifying assumptions should be made explicit, so that one can determine how they falsify the cognitive problem. R. L. Ackoff et al. (1962, p. 117) proposed that the complete justification for such an approximation required a comparison between the "cost" arising from mathematical complexity and the "cost" of lost performance in following a course of action selected by use of the model as compared with a course of action selected by use of a less approximate model.

The conversion of the simplifying assumptions into mathematical forms is conditioned by the choice of variables and the functional form of the model.

This process can be carried out on the basis of common sense, or by reason of more sophisticated theoretical considerations, or by analogy.

The choice of variables leads to the omission of some of them, because their impact on the performance of the model is quite small, and their contribution to the mathematical complexity large. Also, the mathematical characteristics of the variable may be changed to simplify its handling; for example a continuous variable may be treated as discrete, or vice versa. However, the chief problem in the construction of a model is the choice of mathematical function. As a general rule the simplest function is desired. Linear functions are usually the simplest to deal with and nonlinear functions are frequently approximated by linear functions. Furthermore, if the function is not specified by a hypothesis or theory, it has to be determined by empirical search.

An interpreted mathematical model has two fundamental cognitive functions: the hypothetico-deductive and hypothetico-empirical. The hypothetico-deductive function of a model facilitates the easy and accurate deduction of conclusions. The hypothetico-empirical function leads to the concretization of the model, i.e. giving numerical values to the parameters of the model. The process of concretization of the model is based on the theory of statistical estimation. Here the researcher faces a difficult choice between the most efficient method of estimating the parameters of the model and other methods which allow reasonable estimates to be made with a minimum amount of computation. Thus regression procedures are frequently used to estimate relationships, but the appropriateness of such procedures, particularly with respect to socio-economic phenomena, has been called into question by some writers. A discussion of this issue, however, involves a discussion of the whole rationale of statistical methods. The estimation of parameters requires a sound knowledge of the mathematical properties of the different methods of estimation and cannot rely upon intuition and good sense alone.

Models may also be classified as stochastic or deterministic. However, there is a controversy about this classification, a controversy which can be found in almost any field, except where the model can be formulated only in one of the two ways. The main argument for the probabilistic or stochastic approach is that social or economic processes are in fact probabilistic, so that a deterministic process is only a poor substitute. But, as J. Coleman (1964, p. 427) suggests, a deterministic model has some advantages. It can reflect in a more simple manner the same basic process as does the stochastic model. Thus it is possible to treat processes of a far greater degree of complexity than the cumbersome stochastic equation would allow. In addition one of the tendencies in the use of a probabilistic model is to develop a kind of "know-nothing" approach towards the behaviour of the system that the model is intended to reflect. Such a model has few or no definite statements about cause and effect or the relations between variables.

In the realm of social science mathematical models can be classified also into (1) a descriptive or class phenomena model, and (2) a normative or decision model. A descriptive or class phenomena model simply expresses significant causal or structural relationships stripped of the irrelevancies and complexities of the real world. A normative model specifies what ought to happen (or be made to happen) if an optimum (or satisfaction) is desired.

Because economic geography is a discipline predominantly concerned with cognition and not with the strategies of decisions it should concentrate upon the construction of descriptive models as one of the methods allowing the formation of quantitative generalizations, and, in my opinion, a descriptive model can play a more important role in economic geography than can a normative one. Such descriptive models provide a basic framework for a predictive and theoretical science in which the emphasis is placed on empirical statements. The procedure of statistical estimation of the parameters of the model will then allow one to state specific empirical relations.

The main stream of geography is concerned with the collection, classification and ordering of data. This descriptive, old-fashioned view of geography played, in the early stages of the discipline, an important role, but now it is not sufficient. In essence, descriptive economic geography does not focus on the most necessary goals of modern science, namely explanation and prediction. The application of a mathematical model is strictly associated with the role of economic geography as a predictive or theoretical discipline, with emphasis placed on making general statements. In this context an essential point is that a model provides the framework on which we can build a theory, and mathematical models and statistical methods are means to help achieve this end.

In the present part of the paper I wish to discuss gravity and potential models. They consist of simplified assumptions for studying interaction in space, and they are the simplest and most widely used interaction models. Summaries of the literature can be found in a review article by G.A.P. Carrothers (1956) and in the works of W. Isard (1960), G. Olsson (1965) and Z. Chojnicki (1966).

The gravity and potential models should be considered in the first place from the point of view of their expediency for space economics. Such expediency, however, is dependent on the cognitive value of the models, which consists of a proper description of reality. Gravity and potential models can be applied in many fields, for they are important as generalizing descriptions and as elements of various more complex models. According to W. Isard (1960), the models in question are of substantial significance for regional planning, since they complement other analytical and forecasting methods, such as input-output and linear programming, which have gained general application in research.

The models in question do not have uniform constructions, so they will be dealt with separately.

THE GRAVITY MODEL

The initial statement of the gravity model in a form analogous to the basic notions of Newtonian physics should be treated as an attempt to describe stable, structural space relations in the behaviour of populations. This model has purely heuristic meaning, since in the present state of knowledge it seems to be impossible to establish a transformation which would "project" the properties of a physical field into economic space. Such an analogy, however, may be treated as an initial step in formulating a simplified assumption about actual spatial behaviour. The proper gravity model, or better, the gravity model of interaction in space, represents such an assumption. The model focuses on the structural interdependence concerning interaction of masses (populations) and omitts the motivation of individual behaviour. The model is also empirical, because the mechanism connecting distance with the frequency of interaction cannot be explained directly by a centrally symmetric and strictly monotonic function of distance, and because mass must be discriminated in relation to the type of mutual interaction.

These assumptions are formulated as an equation: —

$$I_{ij} = G\,\frac{(w_iM_i)^{\alpha_i}\,(w_jM_j)^{\beta_j}}{D_{ij}^{b}}$$

where

I_{ij} = interaction between places i and j;
M_j = mass of inhabitants in place j;
M_i = mass of inhabitants in place i;
D_{ij} = distance between places i and j;
w_i = weight attached to M_i;
w_j = weight attached to M_j;
G = empirically derived constant;
α_i, β_j, b = empirically derived exponents.

The gravity model formulated as above poses, however, conceptual and technical problems, namely: —

(1) The choice of adequate measures of mass and distance, and

(2) The choice of exponents attached to such variables.

When this equation is taken as the general assumption for investigations concerning interaction with respect to a determined range of empirical phenomena, the problem is placed in the domain of a mathematical model. This enables a statistical estimation of the parameters of the model, once the form of the appropriate function has been established.

It should be noticed that most mathematical models describing quantitative relations between various economic and non-economic magnitudes are the expression of general economic laws. But the lack of an adequate theory makes it impossible to specify general regularities in space economic research. In the case of the gravity model the normal course of establishing a mathematical model is reversed. Starting with a function similar to the notion of gravitation energy one endeavours to determine analytically the nature of the assumptions necessary for investigating economic interrelations in space.

Empirical research on the part played by distance in interaction demonstrates that the function may be approximated by a linear function. Available results of research also demonstrate that the gravity model can be presented as a simple regression of log I_{ij}/M_iM_j as the dependent variable on log D_{ij} as the independent variable. That is, we have: —

$$\log \frac{I_{ij}}{(w_iM_i)^{\alpha_i}\,(w_jM_j)^{\beta_j}} = \log G - b \log D_{ij}^b$$

Such a formulation, which may be called the regression form of the gravity model, enables one to make a statistical estimation of the parameters of the model. Such a statistical analysis is relatively simple, since: —

1) the identification problem does not occur here, i.e. the parameters of the model can be always estimated statistically,

2) estimation may be done by means of the least squares method, although other methods of estimation should not be excluded. It should be stressed that the estimation of the model parameters, formulated above, enables one to determine the exponent of the distance variable, and to verify the model by means of variance analysis. The model thus defined, fulfilling the conditions of verification, may be treated as an empirical regularity, having very definite restrictions in time and space. Those restrictions are determined by the framework of statistical data, within which the parameters of the model have been estimated and verified. Simplicity and easy estimation are basic virtues of the model thus determined.

It must be stated, however, that correct estimation of the parameters of the model is limited, in the first instance, by its assumptions. That is why the cognitive value of the model is dependent on the variables of the model and their conceptual identification, that is to say, on the determination of the types of interaction of mass and distance. This means that the basic gravity model is superimposed upon reality. The appropriateness of this procedure has not been closely evaluated from a methodological point of view, although it is clearly of crucial significance in evaluating the applicability of the model. The difficulty of identifying *a priori* a problem situation to which the gravity model may be applied, is the weakest point in the methodological analysis of the applicability of the model to social and economic phenomena. Thus far, research has largely amounted to the simplified evaluation of the influence of distance and mass, those two last elements being taken as basic factors which determine the quantity interpreted as spatial interaction.

Adequate statistical data, in terms of their amount and precision, are the second essential element in correct estimation. Previous research, especially the investigations of M. Helvig (1964), demonstrate that if the above assumptions are met, essential interrelations describing interaction are detected. One should, however, point to the necessity of keeping a certain balance in determining the degree of detail within the notions of interaction and mass variable. Excessive detail may lead to distortions in the structural unity of the dependence being considered, and it may also cancel the assumption concerning the mass character of the investigated phenomena.

It should also be stressed, that the gravity model cannot be applied correctly to such spatial phenomena or processes which do not correspond to its assumptions, or which are of a more complex character. However, the review of recent research by Z. Chojnicki (1966) has also shown that the gravity model may be used to determine other variables, such as effective distance, social residue etc., which extends the range of its application. Empirical relations or hypotheses obtained on the basis of the model are a step towards setting up an empirical theory of spatial structure.

THE POTENTIAL MODEL

The concept of the potential model is derived from the gravity model. It is represented thus: —

$$V_i = G \sum_{j=1}^{n} \frac{(w_jM_j)^{\beta_j}}{d_{ij}^b}$$

where G, W_j, M_j, β_j, d_{ij} and b are defined as before, and where there are 1, 2, ... n places j, and V_i is the measure of the potential exerted at place i by the n places j. This procedure is an example of the deductive role of a mathematical model. The potential model is concerned with the influence of all masses on one point, while the gravity model concerns interaction of pairs of masses. In the potential model, contrary to the gravity model, the empirical correspondent of the value of interaction, expressed by the potential at a given point, cannot be strictly determined. That is why evaluation of the parameters of this model, and especially of the parameter of distance, is impossible. Thus, the potential model cannot be empirically verified in a direct way. It may, however, be verified indirectly, when determining the degree of correlation between the magnitude of the interpreted potential model and other phenomena, e.g. as the potential of population or of income with other social or economic phenomena.

Criticism of the potential model is based on the fact that the notion of potential cannot have a strict interpretation in relation to the spatial structure of social and economic phenomena, and that it cannot be statistically approximated. This criticism is based on the assertion that the model in question assumes an unlimited spatial continuum, and only describes the tendencies toward spatial equilibrium.

However, such criticism does not seem justified. From the point of view of its logical construction the potential model should be interpreted as a model of measurement, serving as a measurement standard and not as an empirical hypothesis. In relation to spatial research on social and economic facts, in which the actual systems are isolated to a certain degree (countries, continents), the application of the model may be limited to an integrated description of such system. The role of the potential model is based on the fact that each of the elements of the system (i. e. areal unit, region) is characterized in relation to all other elements (and to itself), considered from the point of view of their spatial economic interrelations, measured by distance within the system. The range of empirical interpretations of the model is expressed by the specification of the notions of mass and distance. Thus, the potential model enables one to find a measure of an empirically interpreted mass (of population or income), in a given place, from the point of view of spatial differentation of other masses. When the potential is determined for all masses under consideration, the effect of the situation of each of those masses within the system in question can be taken into account. It follows that the potential model as a research assumption fulfils the postulate of treating the social and economic reality in this spatial aspect as a structural whole.

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